

On the multiplicities of families of complex hypersurface-germs with constant Milnor number

Camille Plénat and David Trotman

December 3, 2011

Abstract. *We show that the possible drop in multiplicity in a polynomial family $F(z, t)$ of complex analytic hypersurface singularities with constant Milnor number is controlled by the powers of t . We prove equimultiplicity of μ constant families of the form $f + tg + t^2h$ if the singular set of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{h = 0\}$.*

1. Background.

Let $F : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function in z_1, \dots, z_n and t .

We study the following conjecture, stated implicitly by Teissier in 1974 in his Arcata survey [18] as well as at the beginning of his Cargèse paper [17], and which implies a parametrised version of Zariski's problem [23] about the topological invariance of the multiplicity (Conjecture 1.2 below).

Conjecture 1.1 (Teissier 1972 [17]). *If for t in some neighbourhood U of 0 in \mathbb{C} , each function $F(., t)$ has an isolated singularity at the origin with the same Milnor number μ , then the functions $F(., t)$ have the same multiplicity at 0 for $t \in U$.*

Teissier made the stronger conjecture at Cargèse in 1972 that μ -constancy implies that the Whitney conditions hold for $(F^{-1}(0), 0 \times \mathbb{C})$. (The same conjecture was made by Lê Dung Trùng and Ramanujam in [11], published in 1976, although submitted in June 1973.) This turned out to be false as first illustrated by the famous examples of Briançon and Speder [5]. Thus Conjecture 1.1 may also be considered as a conjecture of Teissier which remains open. It has two corollaries, as follows.

Conjecture 1.2 (Zariski's problem for families). *Families of complex analytic hypersurfaces with isolated singularities of constant topological type are equimultiple.*

Proof. This would follow from Conjecture 1.1 because the Milnor number is a topological invariant (Milnor [13], Teissier [17]). \square

Conjecture 1.3 *Bekka (C)-regular families of complex hypersurfaces are equimultiple.*

Proof. Use the analogue of the Thom-Mather isotopy theorem for (C)-regularity as proved by Bekka in his thesis [1], together with Conjecture 1.2. \square

Equimultiplicity was established in the case of Whitney regularity for general complex analytic varieties by Hironaka [9], and with a different proof, for Whitney regularity of families of complex analytic hypersurfaces by Briançon and Speder [6]. The proof of Briançon and Speder was first extended to arbitrary complex analytic varieties by Navarro Aznar in [14], and the result is a special case of Teissier's

general characterisation [19] of Whitney regularity in terms of equimultiplicity of polar varieties.

Conjecture 1.3 for the stronger hypothesis of weak Whitney regularity (defined by Bekka and Trotman [2], weak Whitney regularity is weaker than Whitney regularity but stronger than (C) -regularity [3]) was proved directly in 2010 [22] by the second author and Duco van Straten, i.e. weak Whitney regularity implies equimultiplicity for families of complex hypersurfaces.

Conjecture 1.2 is still unproved, as is Conjecture 1.3. It is also unknown whether constant topological type implies (C) -regularity. It was shown recently by Bekka and Trotman [4] that (C) -regularity is in general weaker than weak Whitney regularity for the 1975 quasi-homogeneous examples of Briançon and Speder [5]. No example is currently known of a weakly Whitney regular complex analytic stratification not also satisfying Whitney regularity, while the equivalence of the two conditions has only been proved in the classical case of a family of plane curves, using that weak Whitney regularity implies (C) -regularity [3], which implies topological triviality by [1], and hence constant Milnor number, and Whitney regularity is equivalent to constancy of the Milnor number for families of plane curves [18].

Lê-Saito-Teissier criterion for μ constancy.

According to Lê and Saito [10] and Teissier [17], the constancy of the Milnor number of $F(., t)$ is equivalent to F being a Thom map, i.e. equivalent to the (a_F) condition being satisfied. This can be reformulated as saying that

$$\frac{|F_t|}{||F_z||} \rightarrow 0 \text{ as } (z, t) \rightarrow (0, 0)$$

where F_t is notation for $\partial F / \partial t$, F_z is notation for $(\partial F / \partial z_1, \dots, \partial F / \partial z_n)$, $|\cdot|$ denotes the modulus of a complex number and $||\cdot||$ denotes the hermitian norm on \mathbb{C}^n .

In this paper we use this criterion for constancy of the Milnor number to determine some situations when equimultiplicity holds (Propositions 1.1 and 3.2), and to reduce possible jumps in the multiplicity (Propositions 2.1 and 2.2).

Write $F(z, t) = f(z) + \sum_{k \geq 1} t^k g_k(z)$.

Then $F_t = \sum_{k \geq 1} k t^{k-1} g_k$, and $F_z = f_z + \sum_{k \geq 1} t^k (g_k)_z$.

Due to its upper semicontinuity the multiplicity is non-constant iff $m = m(f) > m_1 = \min_{k \geq 1} m(g_k)$ for all t in some punctured neighbourhood of 0.

Proposition 1.1 *If $F(z, t) = f(z) + tg(z)$ is a 1-parameter family of isolated complex analytic hypersurface singularities whose Milnor numbers are constant, then the multiplicity at 0 of g is greater than or equal to the multiplicity at 0 of f .*

Proof.

Suppose that $m(f) = m > m_1 = m(g)$.

Consider analytic arcs $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$, $\gamma(u) = (z(u), t(u))$, such that $\gamma(0) = 0 \in \mathbb{C}^{n+1}$. We must find an arc γ such that

$$\frac{|F_t(\gamma(u))|}{||F_z(\gamma(u))||} \not\rightarrow 0 \text{ as } u \rightarrow 0.$$

For any analytic function $Q : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$, write $V(Q)$ for the order in u at 0 of $Q \circ \gamma : \mathbb{C} \rightarrow \mathbb{C}$, and for any analytic function $P : \mathbb{C}^n \rightarrow \mathbb{C}$ write $v(P)$ for the order in u at 0 of $P \circ \pi_z \circ \gamma$. We must choose an analytic arc γ such that $V(F_t) - \min\{V(\partial F/\partial z_i)\} \leq 0$.

Now $V(F_t) - \min\{V(\partial F/\partial z_i)\} = v(g) - \min\{V(\partial f/\partial z_i + t\partial g/\partial z_i)\}$.

Let $\gamma(u) = (uz_0, 0)$ where $z_0 \in \mathbb{C}^n - \{0\}$.

Then $V(F_t) - \min\{V(\partial F/\partial z_i)\} = v(g) - \min\{v(\partial f/\partial z_i)\}$. For z_0 sufficiently general, the right-hand side is $v(g) - (v(f) - 1) = m_1 - m + 1 \leq 0$, because $m_1 < m$.

Thus $V(F_t) - \min\{V(\partial F/\partial z_i)\} \leq 0$, contradicting the hypothesis that μ be constant, using the Lê-Saito-Teissier characterisation. \square

Remark 1.2. The result of Proposition 1.1 was discovered by the second author during the academic year 1976-77 and announced in a talk given in March 1977 [18] as one of the weekly A'Campo-MacPherson singularity seminars at the University of Paris 7. The text of this talk was included in the second author's Thèse d'État [21] defended at Orsay in January 1980. The result was rediscovered by Gert-Martin Greuel in 1986 [8], and used to prove Teissier's conjecture in the case of quasi-homogeneous, and semi-quasihomogeneous functions f .

Remark 1.3. Parusinski [16] has proved that a μ -constant family of the form $f + tg$ has constant topological type by integrating an appropriate vector field, with an argument which works for all n . Lê Dung Tràng and Ramanujam [11] proved that for a μ constant family of complex hypersurfaces defined by $\{F(z, t) = 0\}$, the hypersurfaces $\{F(z, t) = 0\} \cap \{\mathbb{C}^n \times \{t\}\}$ have constant topological type when $n \neq 3$.

2. Controlling multiplicity.

Proposition 2.1. *If $F(z, t) = f(z) + tg(z) + t^2h(z)$ is an analytic 1-parameter family of isolated hypersurface singularities with Milnor number μ constant, then the multiplicity at the origin of g is greater than or equal to the multiplicity m at the origin of f , and the multiplicity at 0 of h is greater than or equal to $m - 1$.*

Proof. Because

$$\frac{|F_t|}{\|F_z\|} = \frac{|g + 2th|}{\|f_z + tg_z + t^2h_z\|},$$

it follows that on a generic curve of the form $(uz_0, 0)$ with $z_0 \neq 0$, $V(F_t) - V(F_z) = m(g) - v(f_z) = m(g) - m + 1$. Hence if $m(g) - m + 1 \leq 0$, i.e. $m(g) < m$, we obtain a contradiction to the hypothesis that the Milnor number remains constant.

Thus we obtain that the coefficient g of t has multiplicity $m(g) \geq m = m(f)$.

Suppose that $m(h) \leq m - 2$.

On a generic curve of the form (uz_0, ut_0) , with both $z_0 \neq 0$ and $t_0 \neq 0$, if $\Delta = \frac{|F_t|}{\|F_z\|}$, then $\Delta \sim \frac{|2th|}{\|f_z + t^2h_z\|}$.

Hence $V(\Delta) = 1 + m(h) - \min\{m - 1, 2 + m(h) - 1\} = 1 + m(h) - (m(h) + 1) = 0$.

This again contradicts the hypothesis that the Milnor number of the family $F(., t)$ is constant, proving that $m(h)$ is at least $m - 1$. \square

Now we generalize to arbitrary deformations of f which are polynomial in t .

Proposition 2.2. *If the family $F(z, t) = f(z) + tg_1(z) + t^2g_2(z) + t^3g_3(z) + \dots + t^r g_r(z)$ has constant Milnor number at $(0, t)$ as t varies in a neighbourhood of 0,*

and f has multiplicity m at the origin, then $m(g_1) \geq m$, $m(g_2) \geq m - 1, \dots$, and $m(g_r) \geq m - r + 1$.

Proof.

Here, $V(F_t) - V(F_z) = \min_{k \geq 1} \{(k-1) + v(g_k)\} - V(f_z + \sum_{k \geq 1} t^k (g_k)_z)$, assuming that we are on a generic arc for which there is no cancellation of terms in the expression for F_t .

In particular, $V(F_t) = \min\{m(g_1), m(g_2) + 1, m(g_r) + r - 1\}$, and
 $V(F_z) = \min\{m(f) - 1, m(g_1) - 1 + 1, m(g_2) - 1 + 2, \dots, m(g_r) - 1 + r\}$
 $= \min\{m - 1, V(F_t)\}.$

But the family $F(z, t)$ has constant Milnor number, so that $V(F_t) > V(F_z)$, by the Lê-Saito-Teissier theorem. The conclusion follows. \square

It is interesting to compare the previous result with the following, proved by Greuel in 1986 [8]. Observe the extra restrictions Greuel imposed on the $\{g_i\}$.

Proposition 2.3 (Greuel). *Let $\lambda_j : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ and $g_j : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, $j = 1, \dots, r$, be holomorphic functions such that $\lambda_j \neq 0$ and the initial forms of g_j are \mathbb{C} -linearly independent. Assume that*

$$F(z, t) = f(z) + \sum_{j=1}^r \lambda_j(t) g_j(z)$$

is a μ -constant unfolding of f . Then $\nu(\lambda_j) + m(g_j) > m(f)$ for all $j = 1, \dots, r$, where $\nu(\lambda)$ denotes the order in t of $\lambda(t)$.

3. Obtaining equimultiplicity.

Proposition 3.1. *Let $F(z, t)$ be a μ -constant family of complex hypersurfaces with isolated singularities at $z = 0$ for each t in a neighbourhood of 0, of the form $F(z, t) = f(z) + \sum_{k=1}^r t^k g_k(z)$. Suppose that the tangent cone of f has an isolated singularity at 0. Then the multiplicity $m(F(\cdot, t))$ is constant as t varies in a neighbourhood of 0.*

Proof. Suppose that the tangent cone $\{f_m = 0\}$ has an isolated singularity. Then f is semi-homogeneous, in particular semi-quasihomogeneous, and by work of Varchenko, Greuel [8] and O'Shea [16] independently proved equimultiplicity. The special case of homogeneous f was previously treated by Gabrielov and Koushnirenko [7]. \square

Motivated by the proof of Proposition 2.1, we could study what happens in a family $F(z, t) = f(z) + tg(z) + t^2 h(z)$ with constant Milnor number if we take a more general generic curve of the form $(u^p z_0, u^q t_0)$ with $z_0 \neq 0, t_0 \neq 0$, and where $p \neq q$ and p and q are non-negative integers. This turns out not to be fruitful however.

So we change tactics by choosing an appropriate *non-generic* line segment, whereas the previous results were obtained by choosing suitable *generic* line segments.

Proposition 3.2. *Let $F(z, t)$ be a μ -constant family of complex hypersurfaces with isolated singularities at $z = 0$ for each t in a neighbourhood of 0, of the form $F(z, t) = f(z) + tg(z) + t^2 h(z)$. Suppose that the singular set of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{h = 0\}$. Then the multiplicity $m(F(\cdot, t))$ is constant as t varies in a neighbourhood of 0.*

Proof. By Proposition 2.1, we know that $m(g) \geq m = m(f)$, and that $m(h) \geq m - 1$.

Assume that $m(h) = m - 1$.

For a complex line segment $\gamma(u) = (uz_0, ut_0)$, calculating

$$V(\Delta) = V\left(\frac{|g + 2th|}{\|f_z + tg_z + t^2h_z\|}\right) = V(g + 2th) - \inf\{V(f_{z_i} + tg_{z_i} + t^2h_{z_i})\},$$

we note that if $\inf\{v(f_{z_i})\} > m - 1$ then the tangent cone of $\{f = 0\}$ in \mathbb{C}^n must have a non isolated singularity at 0 and the line segment $\gamma(u) = (uz_0, ut_0)$ must be such that uz_0 lies in the singular locus $\Sigma(f_m)$ of the tangent cone to $\{f = 0\}$. If in addition $\Sigma(f_m)$ is not contained in the tangent cone to $\{h = 0\}$, then $v(h) = m(h)$, which equals $m - 1$ by assumption, and because $m(g) \geq m$ we can choose a generic t_0 in \mathbb{C} so that

$$V(g + th) = V(th) = m,$$

$$\text{and } V(f_z + tg_z + t^2h_z) \geq m.$$

It follows that $V(\Delta) \leq m - m = 0$, which implies a contradiction to the hypothesis of constant Milnor number, by the Lê-Saito-Teissier criterion. \square

Remark 3.3. Similarly to the argument in the previous proof we can obtain a contradiction to the hypothesis of constant Milnor number by the Lê-Saito-Teissier criterion if $m(g) = m$ and $\Sigma(f_m)$ is not contained in the tangent cone to $\{g = 0\}$. Thus if a family $F(z, t) = f(z) + tg(z) + t^2h(z)$ has constant Milnor number, and the singular locus of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{g = 0\}$, then $m(g) \geq m + 1$.

More generally, the same argument shows that if a family $F(z, t) = f(z) + tg_1(z) + t^2g_2(z) + t^3g_3(z) + \dots + t^r g_r(z)$ has constant Milnor number, and the singular locus of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{g_k = 0\}$, then $m(g_k) \geq m - k + 2$.

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Laboratoire d'Analyse, Topologie et Probabilités (UMR 6632 du CNRS),
 Aix-Marseille Université,
 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France.

plenat@cmi.univ-mrs.fr, trotman@cmi.univ-mrs.fr